

CIRCULAR SETS

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Abstract. The replacement of straight line segments by circles transforms convexity into a property deserving some investigation.

Introduction.

It all began while preparing a lecture on Mobius transformations for a Complex Analysis class, as it was noticed that open discs in the plane have the following, convexity-like property: every two points in an open disc lie on a circle contained in that open disc. Then we started wondering what other sets share this property with the open disc and came up with the following definition.

Definition. An infinite subset D of the plane is said to be circular if and only if for every two points P, Q in D there exists a circle C such that P, Q lie on C and C is a subset of D .

At a first glance, it seems that circularity is more flexible than convexity: indeed, while there exist uncountably many circles passing through every two points, there exists only one straight line through them. However, circularity imposes a certain "roundness" on

the circular set, forcing it to be, as we will see, a not too distant relative of the open disc.

Exercise 1. Show that the open annulus, defined by $\{(x,y): R_1^2 < x^2 + y^2 < R_2^2\}$, is not a circular set.

The study and classification of circular sets is a good topic for an undergraduate seminar, as all the needed lemmas and propositions could be considered as exercises in College Topology and Geometry. Such a seminar could provide an early, naive exposure to a number of Mathematical phenomena: classification theorems, using tools from one area (Topology) to get results in another area (Geometry) and the need to rigorously establish the intuitively obvious.

As Professor F. Burton Jones pointed out to us, circular sets can be viewed as a special case of Whyburn's cyclicly connected sets[12], with simple closed curves replaced by circles. In another direction, it is certainly possible to generalize the concept of circularity, by replacing circles by circular arcs or the plane by higher dimensional spaces, for example. However, we will not attempt to do so here; rather, we simply provide the reader with a celebration of the circle in its most natural environment, the Euclidean plane $\mathbf{R}^2$.

Before going further, let us introduce some terminology and recall some basic facts. The circle of radius r centered at M is denoted by $M(r)$. A chord of a convex subset C of $\mathbf{R}^2$ is a straight line segment which contains at least one interior point of C and the endpoints of which are boundary points of C . An open disc is the bounded component of the complement, in $\mathbf{R}^2$, of a circle, while a

closed disc is the topological closure of an open disc. The topological closure, interior and boundary of a set A are denoted by $\overline{A}$, A° and ∂A , respectively. A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded ([7], p. 174). A topological space is totally disconnected if and only if all its components (i.e., maximal connected subspaces) are singletons. The first infinite cardinal number is denoted by $\aleph_0$, hence the cardinality of the Real line $\mathbf{R}$ is denoted by $2^{\aleph_0}$.

Establishing circularity.

Returning to the open disc, how do we verify that it is a circular set? One way to do this is the following: given points P, Q in an open disc D , the center M of a circle C passing through P, Q and lying inside D is found as the intersection of the perpendicular bisector of PQ and the radius of ∂D passing through that point in $\{P, Q\}$ that is closer to ∂D (Fig. 1).

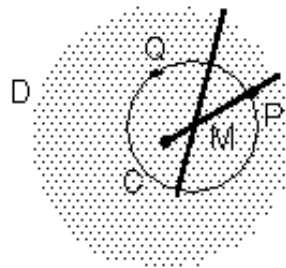
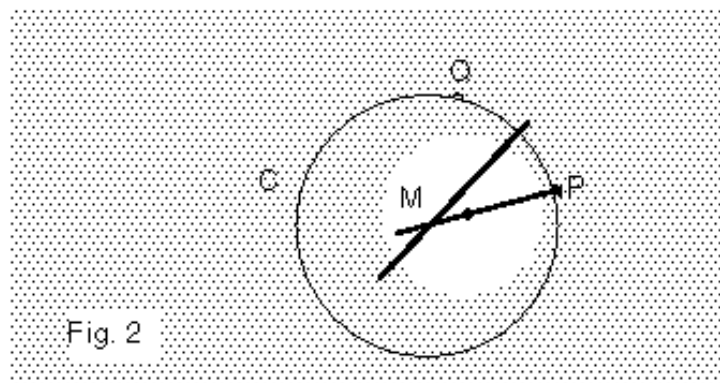


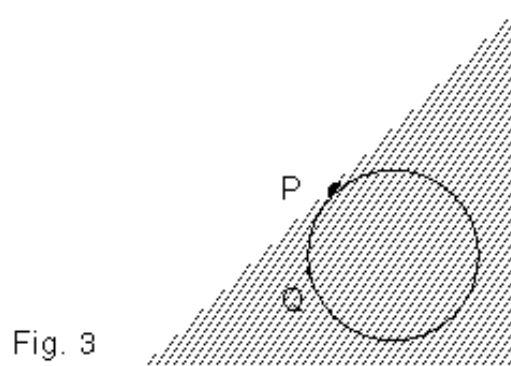
Fig. 1

The same construction that established the circularity of the open disc works also for the closed disc, and, with some caution, for the open half plane (of course this is easier to sell to those who view the open half plane as an open disc of infinite radius). Less

predictably, the construction also works, word by word, for the complement of a closed disc (Fig. 2).



It is easy to see, using the same construction in fact, that adding a single point P to the boundary of an open disc or an open half plane yields two more examples of circular sets: given an interior point Q , there exists precisely one circle passing through P and Q , the circle that is tangent to the boundary at P (Fig. 3).



With the addition of two circular sets that need no introduction, the circle and the plane, we can so far count eight "basic" circular sets; two of them are compact, four are open, six are convex. Some less tractable circular sets can be obtained by "puncturing" any one of the four open basic circular sets; that is, by removing a totally disconnected set, which, as we will see, can always be countably,

and in certain cases uncountably, infinite. We will also see that no circular sets can be obtained by puncturing the other four basic circular sets.

Are there any other circular sets? Intuition suggests that the answer is negative, and much of the rest of the paper is devoted to justify this intuitive answer. More specifically, we classify circular sets as follows.

Theorem. Every circular set is one of the following types:

- (1) A circle.
- (2) A closed disc.
- (3) An open disc with a point on the rim.
- (4) An open disc, possibly punctured.
- (5) The plane, possibly punctured.
- (6) The complement of a closed disc, possibly punctured.
- (7) An open half plane with a point on the edge.
- (8) An open half plane, possibly punctured.

We first classified closed circular sets[0], overestimating their significance. Next we investigated bounded circular sets using only Geometry and Topology and forgetting how circular sets originated in an attempt to explain certain properties of Mobius transformations. So we were rather surprised at how useful the complex mapping $1/z$ turned out to be in reducing the unbounded case to the bounded one and completing the proof of the theorem; but, of course, this was only fitting given the origin of the concept.

Exercise 2. Show that $\mathbf{R}^2$ is the only circular set that is a

group under vector addition. (Hint: observe that the circular set must be dense in $\mathbf{R}^2$ and use the "geometrical" identity

$$\begin{aligned} & \{(x,y): (x-h)^2 + (y-k)^2 = R^2\} + \{(x,y): (x-h)^2 + (y-k)^2 = R^2\} = \\ &= \{(x,y): (x-2h)^2 + (y-2k)^2 \leq 4R^2\} . \end{aligned}$$

Exercise 3. Develop an "algebra of circles" in the spirit of the identity given in Exercise 2.

Compact circular sets.

Although circularity is a purely geometrical concept, we will see that Topology, and in particular compactness, is an indispensable tool for the classification of circular sets. In this section, compactness comes into play by generating some very special circles which, together with some Geometry of course, will help us prove a rather "obvious" result: there are no other compact circular sets than the circle and the closed disc. Not surprisingly, convexity will be part of the game, too.

We begin with a simple geometrical result that explains, for example, why the complement of an open disc is not a circular set.

The Tangent Line Trick. If the straight line segment TN is tangent to a circle C at T , then every circle passing through T and N must also pass through points inside C .

Proof: Arguing by contradiction, let's assume that there exists a circle $M(r)$ passing through T and N that does not go inside C .

This implies that $M(r)$ and C are tangent to each other at T (Fig. 4), therefore TNM is a right triangle with $IMTI = IMNI = r$, contradiction.

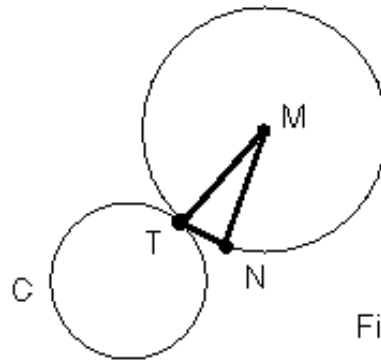


Fig. 4

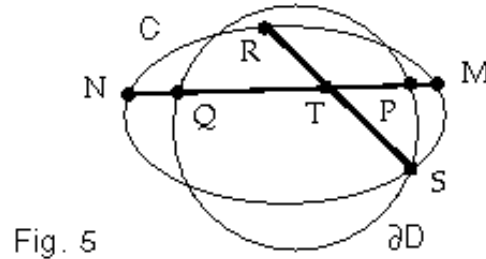
Exercise 4. Suppose that C is a circular set with the following property: for every two points P, Q in C there exists precisely one circle contained in C that passes through P and Q . Show that C is a circle.

The next result is, to some extent, the cornerstone of this paper; although the fact that every two distinct circles have at most two points in common makes it intuitively obvious, we will offer a rigorous proof.

The Chord Lemma. Suppose that the points R, S belong to a closed disc D , with at most one of them on the rim ∂D ; if R, S lie on opposite sides of the chord PQ of D , then every circle passing through R, S must also pass through the interior of PQ .

Proof. Let T be the point of intersection of the straight line segments PQ and RS . Consider a circle C passing through R and

S ; T lies inside C , therefore C intersects the line determined by P , Q at two distinct points M , N . Arguing by contradiction, suppose that $|TM| \geq |TP|$ and $|TN| \geq |TQ|$ (Fig. 5).



At this point, let's recall the following result from high school Geometry: if AB , CD are chords of a circle that intersect each other at T , then $|TA| \times |TB| = |TC| \times |TD|$; see for example [4], p. 8. Applied to the circles ∂D and C , this result yields $|TP| \times |TQ| > |TR| \times |TS|$ (as at least one of R , S lies in the interior of D) and $|TR| \times |TS| = |TM| \times |TN|$, therefore $|TP| \times |TQ| > |TM| \times |TN|$. Since $|TM| \geq |TP|$ and $|TN| \geq |TQ|$ yield $|TM| \times |TN| \geq |TP| \times |TQ|$, we have arrived at a contradiction.

Exercise 5. Given non-collinear points P , Q , R , determine the set of all points S such that there exists a circle C passing through R , S and two distinct points in the interior of PQ .

We are almost ready for the two propositions that characterize the circle and the closed disc as the only compact circular sets; they rely heavily on the following two well known topological theorems, consequences of the preservation of compactness by continuous functions ([7], p. 167) and valid in an arbitrary metric space (M,d) .

The minimum distance theorem. Given two disjoint, non-empty compact sets C_1, C_2 , there exist points $x_0 \in C_1$ and $y_0 \in C_2$ such that $d(x_0, y_0) \leq d(x, y)$ for all $x \in C_1$ and $y \in C_2$.

Proof (outline): The continuous function d maps the compact set $C_1 \times C_2$ on a compact, therefore closed and bounded, subset of $\mathbf{R}$, which has a minimum element t_0 ; x_0, y_0 are now determined by $(x_0, y_0) \in d^{-1}(t_0)$. Another proof can be found in [12], p. 29.

The maximum distance theorem. Given a non-empty compact set C , there exist points x_0, y_0 in C such that $d(x_0, y_0) \geq d(x, y)$ for all x, y in C .

Proof (outline): The continuous function d maps the compact set $C \times C$ on a compact, therefore closed and bounded, subset of $\mathbf{R}$, which has a maximum element t_0 ; x_0, y_0 are now determined by $(x_0, y_0) \in d^{-1}(t_0)$. Another proof can be found in [12], p. 29.

Proposition 1. A circular set C is a circle if and only if it is closed and non-convex.

Proof: It suffices to show that if C is closed, non-convex and circular then it is a circle. We will do that by first obtaining a "special" circle $C' \subset C$ with a chord the interior of which is disjoint from C ("forbidden chord"); the "forbidden chord" will force all points inside $\partial C'$ not to belong to C ("forbidden interior"), and this in turn will inflict the same fate on all points outside of $\partial C'$ ("forbidden exterior"). That is, we will show C to be

a circle by establishing $C = C'$ in three steps.

Step 1 ("forbidden chord"): Since C is not convex, it contains two points A, B such that there exists a point E on the segment AB that does not belong to C . It follows that the sets $AE \cap C$ and $BE \cap C$ are disjoint. As subsets of AB , $AE \cap C$ and $BE \cap C$ are bounded; they are also closed, as intersections of the closed intervals AE, BE with the closed set C . So, $AE \cap C$ and $BE \cap C$ are two disjoint, non-empty, compact subsets of $\mathbf{R}^2$; by the minimum distance theorem, there exist points $P \in AE \cap C$ and $Q \in BE \cap C$ such that no point in the interior of PQ can belong to C . On the other hand, since C is circular, there exists a circle $C' \subset C$ passing through P, Q (Fig. 6); the chord PQ of C' is the "forbidden chord" we were looking for.

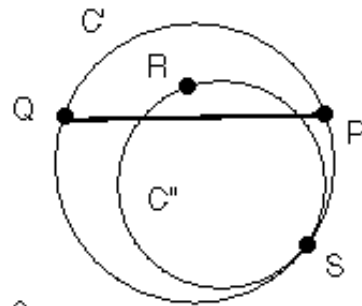


Fig. 6

Step 2 ("forbidden interior"): Suppose that there exists $R \in C$ inside the circle C' . Let S be an arbitrary point on C' , so that R and S lie on opposite sides of PQ (Fig. 6). By circularity of C , there exists a circle $C'' \subset C$ passing through R and S . At this point, the Chord Lemma forces C'' to cross the interior of the "forbidden chord" PQ , which is a contradiction. We conclude that no points inside the circle $C' \subset C$ can belong to C .

Step 3 ("forbidden exterior"): Suppose that there exists $N \in C$

outside of the circle C' . Let TN be tangent to C' at T (Fig. 7).

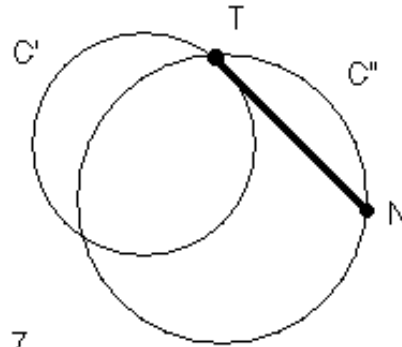


Fig. 7

By circularity of C , there exists a circle $C'' \subset C$ passing through N and T . By the Tangent Line Trick, C'' has to cross into the "forbidden interior" of C' , which is a contradiction. We conclude that no points outside of the circle $C' \subset C$ can belong to C , and this completes the proof.

Proposition 2. A circular set D is a closed disc if and only if it is compact and convex.

Proof: It suffices to prove that every compact, convex, circular set D is a closed disc. Again, everything will start with a "special" circle. In the course of the proof, the statement that "a chord of maximum length is a diameter" will be given a new meaning.

Since D is compact and convex, the maximum distance theorem implies the existence of a chord PQ of D of maximum length. By circularity of D , there exists a circle $M(r) \subset D$ passing through P and Q . By convexity of D , the closed disc D' defined by $\partial D' = M(r)$ is a subset of D .

Observe that PQ is a diameter of $M(r)$; for, if this is not the case, then we obtain the contradiction $|PR| > |PQ|$, where PR is the

diameter of $M(r)$ that ends at P (Fig. 8).

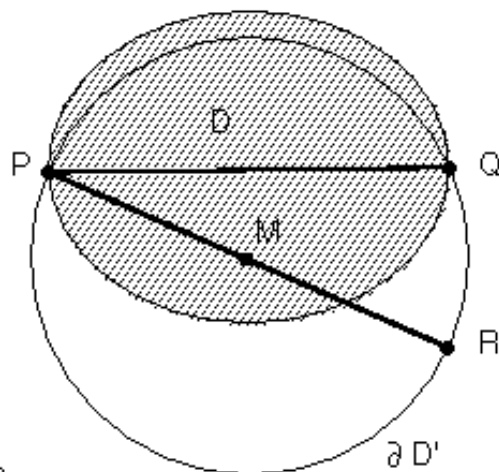


Fig. 8

Suppose now that there exists $N \in D \setminus D'$, and let S, T be the points of intersection of $M(r)$ with the line passing through N and M (Fig. 9);

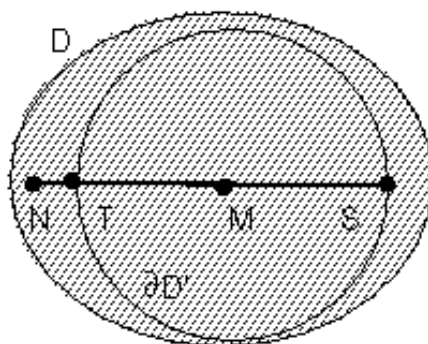


Fig. 9

it follows that $|NS| > |TS| = 2r = |PQ|$, which is a contradiction. We conclude that $D \setminus D' = \emptyset$, therefore $D = D'$ is indeed a closed disc.

Exercise 6[0]. Show that the only closed circular sets are circles, closed discs and the plane. (Hint: it suffices to show, using for example Exercise 5, that the only closed, convex, unbounded circular set is the plane.)

Bounded circular sets.

We have seen that a compact circular set is either a closed disc (in case it is convex) or a circle (in case it is not convex). Since closures of bounded subsets of $\mathbf{R}^2$ are compact, the preservation of circularity by the closure operation would lead to a better understanding of bounded circular sets. Although closures of circular sets do not have to be circular, as the example of the open half plane shows, it turns out that exactly as much as we need is true.

The Closure Lemma. If a circular set is bounded, then its closure is also a circular set.

Proof: Let B be a bounded circular set. Since each point in $\overline{B}$ is the limit of a sequence of points of B , it seems that for every two points P, Q in $\overline{B}$ we should be able to "construct" a circle passing through P, Q as a "limit" of circles in B ; such a "limit circle" would hopefully be a subset of $\overline{B}$. All this does actually work, thanks to the boundedness of B . The details are as follows.

Let $\{P_n\}, \{Q_n\}$ be sequences of points in B converging to P and Q , respectively. Since B is a circular set, there exist circles $M_n(r_n) \subset B$ passing through P_n and Q_n . Since B is bounded, the subsets $\{M_n\}$ of $\mathbf{R}^2$ and $\{r_n\}$ of $\mathbf{R}$ are also bounded, hence we may assume, by passing to subsequences if necessary (Bolzano-Weierstrass property), that there exist $M \in \mathbf{R}^2$ and $r \in \mathbf{R}$ such that $\lim_{n \rightarrow \infty} |M - M_n| = \lim_{n \rightarrow \infty} |r - r_n| = 0$. Taking limits on both sides of the

triangle inequalities

$|MP| \leq |MM_n| + |M_nP|$ and $|M_nP| \leq |M_nM| + |MP| + |PP_n|$,
 we conclude, in view of $\lim_{n \rightarrow \infty} |MM_n| = \lim_{n \rightarrow \infty} |PP_n| = 0$ and $\lim_{n \rightarrow \infty} |M_nP| = r$,
 that $|MP| = \lim_{n \rightarrow \infty} |M_nP| = r$; similarly, $|MQ| = r$, therefore the
 "limit circle" $M(r)$ passes through P and Q .

It remains to show that $M(r) \subset \overline{B}$. Let $S \in M(r)$; we need to
 show $S \in \overline{B}$. Choose points $S_n \in M_n(r_n) \subset B$ so that the vectors MS

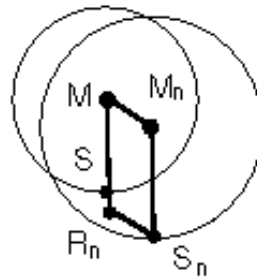


Fig. 10

and M_nS_n have the same direction (Fig. 10). Let $MM_nS_nR_n$ be the
 parallelogram determined by M , M_n and S_n . Taking limits on both
 sides of the triangle inequality $|SS_n| \leq |R_nS_n| + |R_nS|$, we conclude,
 in view of $|R_nS_n| = |MM_n|$ and $|R_nS| = |MS| - |M_nS_n| = |r - r_n|$, that
 $\lim_{n \rightarrow \infty} |SS_n| = 0$; that is, $S \in \overline{B}$.

Exercise 7. Prove or disprove: the interior of a circular set is
 either empty or circular.

In conjunction with Propositions 1 and 2, the Closure Lemma
 implies that the closure of a bounded circular set B is either a
 circle (in which case B itself is a circle) or a closed disc. The
 second case is investigated in the next two Propositions; those
 indicate, among other things, why the removal of a singleton from a
 closed disc or a non-trivial connected set from an open disc

destroys circularity.

Proposition 3. Let B be a circular set whose closure, $\overline{B}$, is a closed disc. If $B \cap \partial D \neq \emptyset$, then either $B = D$ (i.e., B is a closed disc) or $B = D^\circ \cup \{P\}$ (i.e., B is an open disc with a point on the rim).

Proof: This result relies on the following simple remark, the verification of which is left to the reader: the only circle that passes through two distinct points lying on a circle C without going through the exterior of C is C itself.

Now we show that the presence of a single point of B on ∂D forces D° to be a subset of B . Let $P \in B \cap \partial D$ and suppose that there exists $Q \in D^\circ \setminus B$. By our introductory remark, any circle that passes through P and Q and is a subset of D cannot have points in common with ∂D other than P , therefore it has to be the unique

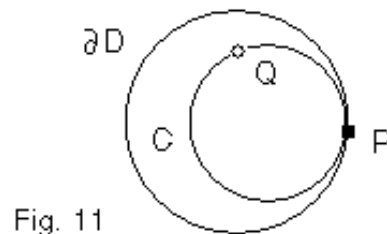


Fig. 11

circle C that is tangent to ∂D at P and passes through Q (Fig. 11). It follows that no point $Q' \neq P$ of C can belong to B : indeed, if $Q' \in B$, then there exists a circle $C' \subset B \subset D$ passing through P and Q' ; arguing as above, we obtain $C' = C$, hence $Q \in B$, which is impossible. The existence of this "forbidden circle" C in D° , together with $\overline{B} = D$ and circularity of B , leads to a contradiction: no circle can pass through a point of B inside C and a point of B

outside of C without intersecting C twice. We conclude that $P \in B \cap \partial D$ implies $D^0 \subset B$.

By the introductory remark again, it follows from the circularity of B and $\overline{B} = D$ that either $B \cap \partial D = \{P\}$ or $\partial D \subset B$; in view of $D^0 \subset B$ and $\overline{B} = D$, these conditions yield $B = D^0 \cup \{P\}$ and $B = D$, respectively. The proof is complete.

Exercise 8. Assume that a circular set B has the following property: there exists $P \in B$ such that for each $Q \in B \setminus \{P\}$ there exists a unique circle $C_Q \subset B$ passing through P and Q . Show that B is either a circle or a closed disc or an open disc with a point on the rim or an open half plane with a point on the edge.

Proposition 4. Let B be a circular set whose closure, D , is a closed disc. If $B \cap \partial D = \emptyset$ then either $B = D^0$ (i.e., B is an open disc) or $B = D^0 \setminus E$ with E a totally disconnected set (i.e., B is what we call a punctured open disc).

Proof: The conditions $\overline{B} = D$ and $B \cap \partial D = \emptyset$ yield $B \subset D^0$, therefore it suffices to prove that $D^0 \setminus B$ is totally disconnected. That is, it suffices to prove that there are no non-trivial connected subsets of $D^0 \setminus B$. Intuitively, it is obvious that the existence of such connected subsets would be incompatible with the circularity of B ; formally, their existence is ruled out by first "stretching" them into straight line segments.

So, we must show first that $D^0 \setminus B$ contains no straight line segments, or, equivalently, that B intersects all straight line segments $J \subset D$. This is easily done, as the condition $\overline{B} = D$ yields

pairs of points of B lying on opposite sides of J and arbitrarily close to its middle; $B \cap J \neq \emptyset$ is now enforced by the circularity of B (Fig. 12).

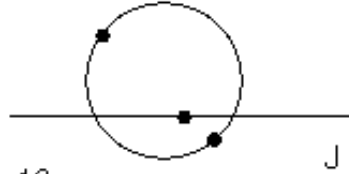


Fig. 12

Now we can show that every non-trivial subset G of $D^0 \setminus B$ fails to be connected and complete the proof. Let P, Q be two distinct points in $G \subset D^0 \setminus B$. By the previous paragraph, there exist points R, S of B between P, Q and on the extension of PQ beyond Q , respectively (Fig. 13).

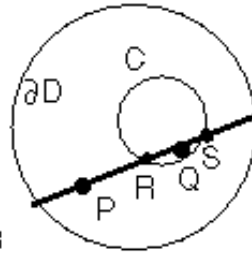


Fig. 13

Let U_1, U_2 be the intersections of $D^0 \setminus B$ with the bounded and unbounded components of $\mathbf{R}^2 \setminus C$, respectively, where $C \subset B$ is any circle passing, by circularity of B , through R and S . Since $C \subset B$, $\{U_1, U_2\}$ provides a "disconnection" of P, Q in $D^0 \setminus B$; that is, U_1 and U_2 are open in $D^0 \setminus B$ (subspace topology), with $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2 = D^0 \setminus B$, $P \in U_2$ and $Q \in U_1$. It follows that the sets $G \cap U_1, G \cap U_2$ are non-empty ($P \in G \cap U_2$, $Q \in G \cap U_1$), open in G (subspace topology), disjoint (by $U_1 \cap U_2 = \emptyset$) and with union equal to G , as $(G \cap U_1) \cup (G \cap U_2) = G \cap (U_1 \cup U_2) = G \cap D^0 \setminus B = G$; that is, G is not connected.

Remark. In Proposition 4, the total disconnectedness of E is a necessary, but not sufficient, condition for the circularity of $D^0 \setminus E = B$; this important issue will be discussed later.

We conclude by noticing that, in view of Propositions 1 through 4, there exist precisely four types of bounded circular sets, the ones given by the Theorem: the circle, the closed disc, the open disc with a point on the rim and the possibly punctured open disc.

Proof of the Theorem.

It is time now to complete the proof of the Theorem by determining the unbounded circular sets. While it might be intuitively clear that there are no unbounded circular sets other than those listed in the Theorem, our proof is unexpectedly indirect.

We first notice, arguing exactly as in the proof of Proposition 4, that if the unbounded circular set B is dense in $\mathbf{R}^2$, then either $B = \mathbf{R}^2$ or $B = \mathbf{R}^2 \setminus E$, where E is totally disconnected; that is, B is the possibly punctured plane.

From here on, we assume that the unbounded circular set B is not dense in $\mathbf{R}^2$. Therefore, there exist holes in the plane away from B ; any one of these holes can be used as a "chemical pool" where B will be immersed for a "litmus test" with three possible outcomes: possibly punctured open half plane, open half plane with a point on the edge and possibly punctured complement of a closed disc.

More mathematically, B will be mapped, by a map equal to its own inverse, onto a bounded circular set inside an open disc disjoint from B ; the same mapping will take the bounded circular set back onto B . Since we know the possibilities for the bounded circular set, we also know its image, which is none other than B ! Two mappings are available to do all these wonderful things: geometric inversion, as described in [4], and the closely related Mobius transformation $1/z$, which we briefly describe here for the sake of completeness.

Identifying points $(x,y) \in \mathbf{R}^2 \setminus \{(0,0)\}$ with $z = x + iy$, a mapping $l: \mathbf{R}^2 \setminus \{(0,0)\} \rightarrow \mathbf{R}^2 \setminus \{(0,0)\}$ is defined via $l((x,y)) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$,

that is, $l(z) = 1/z$. It is easy to see that $l(z)$ is a continuous, one-to-one function from $\mathbf{R}^2 \setminus \{(0,0)\}$ onto itself that sends points far away from $(0,0)$ close to $(0,0)$ (and vice-versa) and preserves the unit circle.

Following [3], we observe that $l(z)$ transforms the curve $a(x^2 + y^2) + bx + cy + d = 0$, where $b^2 + c^2 > 4ad$, into the curve $d(x^2 + y^2) + bx - cy + a = 0$. It follows that:

- (1) Lines ($a = 0$) passing through $(0,0)$ ($d = 0$) are transformed into lines passing through $(0,0)$.
- (2) Circles ($a \neq 0$) passing through $(0,0)$ ($d = 0$) are transformed into lines not passing through $(0,0)$, and vice-versa.
- (3) Circles ($a \neq 0$) not passing through $(0,0)$ ($d \neq 0$) are transformed into circles not passing through $(0,0)$.

Two other properties of $l(z)$ useful for our purposes are that $l(z)$ is its own inverse (as $l(l(z)) = 1/(1/z) = z$) and $l(z)$ preserves total disconnectedness. The latter is true of all topological

homeomorphisms and should be contrasted to the fact that continuous images of certain totally disconnected spaces, such as the Cantor Set, range over all compact metric spaces--see [12], p.67.

Now we can proceed into the proof of the Theorem. Since the unbounded circular set B is not dense in $\mathbf{R}^2$, there exists a closed disc $D_1 \subset \mathbf{R}^2 \setminus B$; by an appropriate change of the coordinate system, we may also assume that ∂D_1 is the unit circle, so that $I[\partial D_1] = \partial D_1$. Since $I(z)$ preserves circles not passing through $(0,0)$, the condition $B \subset \mathbf{R}^2 \setminus D_1$ implies that $I[B]$ is a circular set; and, since $I(z)$ maps points outside of ∂D_1 inside ∂D_1 , $B \subset \mathbf{R}^2 \setminus D_1$ yields $I[B] \subset I[\mathbf{R}^2 \setminus D_1] \subset D_1$, therefore $I[B]$ is a bounded circular set.

Since no element of $\mathbf{R}^2 \setminus \{(0,0)\}$ is mapped to $(0,0)$, $(0,0) \notin I[B]$. Now B is unbounded, so it contains points arbitrarily far from $(0,0)$; $I(z)$ maps these points arbitrarily close to $(0,0)$, thus $(0,0) \in \overline{I[B]}$. We conclude that the bounded circular set $I[B]$ is not closed; therefore, in view of our results on bounded circular sets and $I[B] \subset D_1$, there exists closed disc $D_2 \subset D_1$ such that either $I[B] = D_2^0 \cup \{P\}$ with $P \in \partial D_2$ (in which case $(0,0) \in \partial D_2 \setminus \{P\}$) or $I[B] = D_2^0 \setminus E$, where E is a (possibly empty) totally disconnected set (with $(0,0) \in \partial D_2$ or $(0,0) \in E$). Consequently, there are precisely three cases to investigate:

Case 1: $I[B] = D_2^0 \cup \{P\}$, $P \in \partial D_2$, $(0,0) \in \partial D_2 \setminus \{P\}$.

Since $(0,0) \in \partial D_2$, $I[\partial D_2]$ is a straight line with $(0,0) \notin I[\partial D_2]$. By continuity of $I(z)$, the image of every straight line segment inside ∂D_2 is a connected set (path), forcing the images of the endpoints to lie on the same side of $I[\partial D_2]$ (otherwise $\mathbf{R}^2 \setminus I[\partial D_2]$ would be connected). Since points away from $(0,0)$ (and outside of

∂D_2 , in particular) are mapped close to $(0,0)$, we conclude that the exterior of ∂D_2 is mapped onto that side of $I[\partial D_2]$ that contains $(0,0)$; therefore, D_2^0 is mapped onto that side of $I[\partial D_2]$ that does not contain $(0,0)$ (Fig. 14).

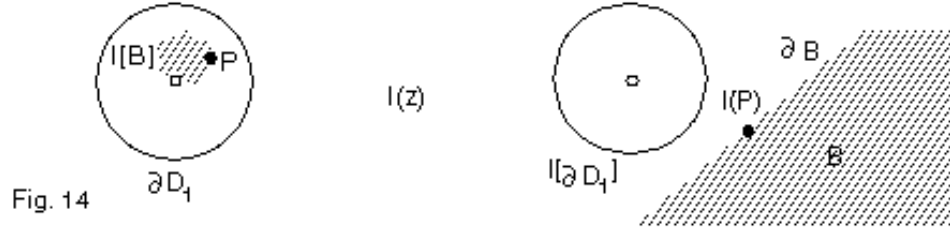


Fig. 14

It follows that $B = I[I[B]] = I[D_2^0] \cup I(P) = (I[D_2])^0 \cup I(P)$, with $I(P) \in I[\partial D_2]$; that is, B is an open half plane with a point on the edge.

Case 2: $I[B] = D_2^0 \setminus E$, $(0,0) \in \partial D_2$, $E \subset D_2^0$, E empty or totally disconnected.

This case is very similar to Case 1, with $I[\partial D_2]$ a straight line and $B = I[I[B]] = I[D_2^0] \setminus I[E] = (I[D_2])^0 \setminus I[E]$, where $I[E]$ has to be empty or totally disconnected. We conclude that B is an open half plane, possibly punctured.

Case 3: $I[B] = D_2^0 \setminus E$, $(0,0) \in E \subset D_2^0$, E (possibly equal to $\{(0,0)\}$) totally disconnected.

As in Case 2, $B = I[I[B]] = (I[D_2])^0 \setminus I[E \setminus \{(0,0)\}]$, with $I[E \setminus \{(0,0)\}]$ empty or totally disconnected. This time, however, $(0,0) \notin \partial D_2$ implies that $I[\partial D_2]$ is a circle; and, arguing as in Case 1, we conclude, in view of $(0,0) \in D_2^0$, that D_2^0 is mapped onto the exterior of the circle $I[\partial D_2]$ (Fig. 15).

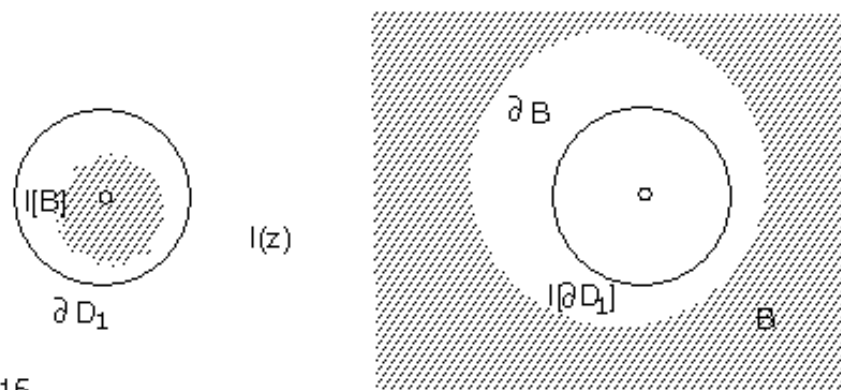


Fig. 15

Therefore, B is the possibly punctured complement of a closed disc.

Our constructive method has automatically ruled out the existence of any unbounded circular sets other than the ones given by the Theorem; put together with the results on bounded circular sets, this completes the proof of the Theorem.

To "complete the circle", we would like to indicate how our investigation of the Möbius transformation $I(z)$ led to the notion of circularity.

Exercise 9. Using the circularity of the open disc (rather than continuity of $I(z)$ and connectedness), show that $I(z)$ maps the interior of a disc D that does not contain $(0,0)$ onto the set of points that lie inside the circle $I[\partial D]$. (Hint: rather than connecting two points of D° with a straight line segment, use a circle passing through them and contained in D° .)

Punctured circular sets.

We have seen already that a necessary condition for preservation of circularity after removing a subset E of an (open) circular set B (with $\overline{B \setminus E} = \overline{B}$) is that E is totally disconnected. Here we discuss the sufficiency of total disconnectedness, providing examples that show how circularity can be preserved by the removal of "large" sets or destroyed by the removal of "small" sets and lead to a research level question. Our examples concern the unit open disc U , their extension to other open circular sets being obvious.

Example 1: Circularity is preserved by the removal of any countable set.

Proof: Let P, Q be any two points in $U \setminus \{R_n\}$, where $\{R_n\}$ is the removed countable subset of U . Since U is open, there exist uncountably many circles inside U passing through P and Q , determined by an open interval MN of "permissible centers" along the bisector of PQ ; more specifically, M and N are defined as the centers of the two circles that pass through P and Q and are tangent to ∂U (Fig. 16).

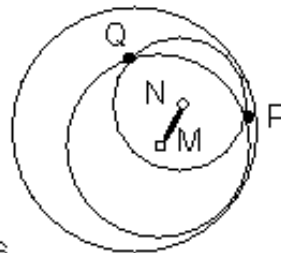


Fig. 16

Since at most one of these circles can pass through each of the

R_n 's, there still exist uncountably many circles in $\mathbf{U} \setminus \{R_n\}$ passing through P and Q , hence $\mathbf{U} \setminus \{R_n\}$ is a circular set. (The same argument shows that the removal of any set of cardinality less than $2^{\aleph_0}$ preserves circularity.)

Exercise 10. Investigate the relation between the circle constructed in Fig. 1 and the two circles (centered at M and N) of Fig. 16.

Example 2: The removal of a totally disconnected set can indeed destroy circularity.

Construction: Let E be the set of all points in $\mathbf{U}$ both coordinates of which are irrational. Any two distinct points $(s_1, s_2), (t_1, t_2) \in E$ such that, say, $s_1 \neq t_1$, are "separated" by the vertical line $x = r_1$, where r_1 is any rational number between s_1 and t_1 ; therefore, E is a totally disconnected set.

We prove that $\mathbf{U} \setminus E$ is not circular by observing that it does not contain any circles at all. Indeed, if C is a circle contained in $\mathbf{U} \setminus E$, then both $\text{Proj}_X(C)$ and $\text{Proj}_Y(C)$ (the projections of C on the coordinate axes) must be countable sets: if, for example, $\text{Proj}_X(C)$ is uncountable, then there must exist uncountably many points on C with an irrational x -coordinate, which forces their y -coordinate to be rational; that in turn would force, by the pigeon hole principle, uncountably many points on C to share the same y -coordinate, which is clearly impossible. We end up having an uncountable set, C , contained in a countable set, $\text{Proj}_X(C) \times \text{Proj}_Y(C)$, contradiction.

Exercise 11. Show that every circle must contain uncountably many points both coordinates of which are irrational.

Exercise 12. Exhibit examples of circles containing precisely 0, 1, 2, ... (go as far as you can) points both coordinates of which are rational.

Example 3: It is possible to preserve circularity by removing a dense, uncountable set.

Construction: This example is built on Example 1. First we fix a countable set $\{R_n\}$ that is dense in $\mathbf{U}$, then we extend each R_n to an one-dimensional Cantor set K_n of arbitrary direction, so that $R_n \subset K_n \subset \mathbf{U}$; of course, $E = \bigcup K_n$ is a dense, uncountable subset of $\mathbf{U}$. (In fact, it is not difficult to select the K_n 's in such a way that every open subset of $\mathbf{U}$ will have $2^{\aleph_0}$ points in common with E .)

Given two points $P, Q \in \mathbf{U} \setminus E$ and an integer n , we define two functions $f_{1,n}, f_{2,n}$ from K_n into the open interval of permissible centers (Example 1) as follows: let S be the point of tangency of a circle C passing through P, Q and tangent to the support line of K_n , with its center as close as possible to that line; now, with $A_1, A_2 \in K_n$ lying on both sides of S , set $f_{1,n}(A_1), f_{2,n}(A_2)$ to be the centers of the circles C_1, C_2 that pass through P, Q, A_1 and P, Q, A_2 , respectively (Fig. 17). (We may assume, without loss of generality, that S lies between the endpoints of K_n and that the circles C_1, C_2 are subsets of $\mathbf{U}$; indeed, each of $\text{dom}(f_{1,n}) = \emptyset$, $\text{dom}(f_{2,n}) = \emptyset$ and $\text{dom}(f_{1,n}) \cup \text{dom}(f_{2,n}) \neq K_n$ is possible.)

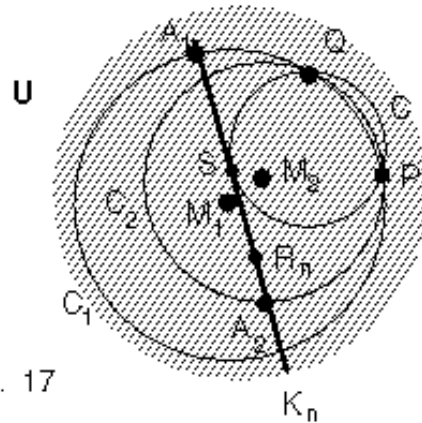


Fig. 17

$$f_{1,n}(A_1) = M_1$$

$$f_{2,n}(A_2) = M_2$$

It is clear that the functions $f_{1,n}$ and $f_{2,n}$ are continuous and monotone. After some use of Analytic Geometry, it can be shown that $f_{1,n}$ and $f_{2,n}$ satisfy the Lipschitz condition, hence they are absolutely continuous; it follows, since monotone, absolutely continuous functions preserve sets of Lebesgue measure zero[10] and $m(K_n) = 0$, that $m(f_{1,n}[K_n]) = m(f_{2,n}[K_n]) = 0$. (Notice at this point that there exist monotone, continuous functions--such as the Cantor ternary function[10], mapping the Cantor Set onto $[0,1]$ --that do not preserve sets of Lebesgue measure zero.) Countable additivity of Lebesgue measure yields $m(\bigcup (f_{1,n}[K_n] \cup f_{2,n}[K_n])) = 0$, therefore $MN \setminus \bigcup (f_{1,n}[K_n] \cup f_{2,n}[K_n])$ is non-empty. It is easy to see now that any circle that has its center on $MN \setminus \bigcup (f_{1,n}[K_n] \cup f_{2,n}[K_n])$ and passes through P, Q has to be a subset of $U \setminus E$; we conclude that $U \setminus E$ is a circular set.

Example 4: It is possible to destroy circularity by removing a nowhere dense, totally disconnected set of planar Lebesgue measure zero.

Construction: Consider $E = \{(x,0): x \text{ irrational, } -1 < x < 1\}$, an "irrational diameter" of $\mathbf{U}$, which certainly satisfies all three conditions. We claim that $\mathbf{U} \setminus E$ is not circular: indeed, any circle $C \subset \mathbf{U} \setminus E$ passing through $(0, -0.5)$ and $(0, y)$, where $0 < y < 1$, must also pass through points $(x_1, 0), (x_2, 0)$, where x_1, x_2 are rational; since there exist only countably many such pairs, as opposed to uncountably many points $(0, y)$ with $0 < y < 1$, $\mathbf{U} \setminus E$ is not circular.

One wonders, in relation to Example 4, whether $\mathbf{U} \setminus F$ fails to be circular whenever $F \subset J \subset \mathbf{U}$, where J is a straight line segment and F has positive Lebesgue measure; in view of Fubini's Theorem's converse ([8], p. 54), an affirmative answer to this question would lead to a negative answer to the following open problem, largely motivated by Example 3.

Question[1]. Is it possible to have a punctured circular set $\mathbf{U} \setminus E$ where E has positive planar Lebesgue measure?

More information on this question and its category version can be found in [1].

Exotic circular sets.

In a conversation following a seminar talk, W. W. Comfort asked whether there exists a circular set E and a point $P \in E$ such that for all other points $Q \in E$ there exist precisely two circles $C_{1,Q}$,

$C_{2,Q} \subset E$ passing through P, Q . It follows from the Theorem that such a circular set would have to be a punctured circular set; but, how does one puncture the open disc or the plane to get it? Does such a set have to exist, after all?

In [5], Fred Galvin answered these questions using the axiom of choice and the method of transfinite induction to construct a circular set substantially more complicated than the ones presented so far, a set that is impossible to "draw". His example is included here with his permission and could serve as an intuitive introduction to transfinite induction for the readers, who are challenged to answer Comfort's question in a "down to earth" way, especially if they do not trust set-theoretic methods!

Example 5 (as in [5]). Given a cardinal number κ such that $2 \leq \kappa \leq 2^{\aleph_0}$, there exists a circular set E_κ such that for every two distinct points P, Q of E_κ there exist precisely κ circles passing through P, Q and contained in E_κ .

Construction: We recursively construct a "lean" circular set $E_\kappa = \bigcup \{C_\alpha : \alpha < 2^{\aleph_0}\}$ built of "good" circles C_α and a set $Z = \{z_\alpha : \alpha < 2^{\aleph_0}\}$ as a "trap" for certain "bad" circles as follows. Let $\{K_\alpha : \alpha < 2^{\aleph_0}\}$ be an enumeration of all the circles in $\mathbf{R}^2$ and $\{(x_\alpha, y_\alpha) : \alpha < 2^{\aleph_0}\}$ be an enumeration of all pairs of distinct points of $\mathbf{R}^2$, each pair being repeated κ times (the latter is possible in view of $\kappa \times 2^{\aleph_0} = 2^{\aleph_0}$).

Assume (transfinite induction step) that distinct circles C_β and points z_β have been chosen for all ordinal numbers $\beta < \alpha$, with

$\alpha < 2^{\kappa_0}$. Let $D_\alpha = \bigcup \{C_\beta \cap C_\gamma : \beta < \gamma < \alpha\}$ be the set of intersection points of all the previously chosen circles, and $Z_\alpha = \{z_\beta : \beta < \alpha\}$, so that

$$|D_\alpha| \leq |\alpha| + \kappa_0 < 2^{\kappa_0} \quad \text{and} \quad |Z_\alpha| \leq |\alpha| < 2^{\kappa_0}.$$

Now, if $\{x_\alpha, y_\alpha\} \cap Z_\alpha \neq \emptyset$ or $|\{\beta < \alpha : \{x_\alpha, y_\alpha\} \subset C_\beta\}| = \kappa$ (i.e., if the pair of points presently under investigation intersects the "bad" set or is found to be the intersection of κ "good" circles already), then we bypass $\{x_\alpha, y_\alpha\}$; that is, we simply choose a circle $C_\alpha \notin \{C_\beta : \beta < \alpha\}$, subject only to the conditions $C_\alpha \cap D_\alpha = \emptyset$ (to avoid accidentally passing another circle through some pair of points which has already met its quota) and $C_\alpha \cap Z_\alpha = \emptyset$ (to stay away from the "bad" set). The existence of such a C_α is guaranteed by $|D_\alpha \cup Z_\alpha| < 2^{\kappa_0}$ and $|\beta| < 2^{\kappa_0}$ (we can argue as in Example 1).

If, however, $\{x_\alpha, y_\alpha\} \cap Z_\alpha = \emptyset$ (i.e., both x_α and y_α are entitled to membership in E_κ) and $|\{\beta < \alpha : \{x_\alpha, y_\alpha\} \subset C_\beta\}| < \kappa$ (i.e., not enough "good" circles have passed through x_α, y_α yet), then we choose a circle $C_\alpha \notin \{C_\beta : \beta < \alpha\}$ so that $\{x_\alpha, y_\alpha\} \subset C_\alpha$ (to raise the number of "good" circles passing through x_α, y_α by one and, at the same time, guarantee inclusion of x_α, y_α in E_κ) and, again, $C_\alpha \cap (D_\alpha \setminus \{x_\alpha, y_\alpha\}) = \emptyset$ and $C_\alpha \cap Z_\alpha = \emptyset$. The latter condition implies that if $\beta < \alpha$ then $z_\beta \notin C_\alpha$.

In both cases we choose z_α as follows. If $K_\alpha \notin \{C_\beta : \beta \leq \alpha\}$ (i.e., if the α -th circle has not yet been chosen as a "good" circle), then we choose $z_\alpha \in K_\alpha \setminus \bigcup \{C_\beta : \beta \leq \alpha\}$, preventing K_α from ever being used as a "good" circle; such a z_α exists because no circle can be covered by less than 2^{κ_0} circles other than itself. If

$K_\alpha \in \{C_\beta : \beta \leq \alpha\}$, then we pick an arbitrary $z_\alpha \in \mathbf{R}^2 \setminus \bigcup \{C_\beta : \beta \leq \alpha\}$, still keeping z_α away from E_κ , in the sense that $\beta \leq \alpha$ yields $z_\alpha \notin C_\beta$. For the first steps of the induction, in particular, we can choose an arbitrary circle $C_0 \neq K_0$ so that $\{x_0, y_0\} \subset C_0$ and an arbitrary $z_0 \in K_0 \setminus C_0$, then a circle $C_1 \neq C_0$ so that $\{x_1, y_1\} \subset C_1$, etc.

After the "end" of this process and the formation of E_κ and Z , it is clear, in view of the preceeding two paragraphs, that $E_\kappa \cap Z = \emptyset$. Given two points P, Q of E_κ , set $H = \{\eta : \eta < 2^{\aleph_0}, \{x_\eta, y_\eta\} = \{P, Q\}\}$, so that $|H| = \kappa$ and $\{x_\eta, y_\eta\} \cap Z_\eta = \emptyset$ for all $\eta \in H$. It follows that, for each $\eta \in H$, a new circle C_η passing through P, Q is added to E_κ , unless of course $|\{\eta < \xi : \{x_\xi, y_\xi\} \subset C_\eta\}| = \kappa$ for some $\xi \in H$; either way, after κ trials, we know that there exist at least κ circles passing through P and Q and contained in E_κ . On the other hand, there cannot be more than κ such circles: by construction, E_κ contains no other circles than the C_α 's and it is not possible for more than κ C_α 's to pass through any two points of E_κ . We have just proved that E_κ has the desired property.

Exercise 13. What happens to the construction of E_κ when $\kappa = 1$? (Compare with Exercise 4.)

Exercise 14[5]. Modify the construction of E_κ to obtain the additional property $|C \cap E_\kappa| \leq 3$ for all circles $C \subset \mathbf{R}^2$. (Hint: notice that $E_\kappa = \mathbf{R}^2 \setminus Z$ and make sure that no 4 distinct points of Z lie on a circle.)

Another look at the Theorem.

The reader has probably noticed by now certain surprises and limitations associated with the Theorem and its proof. For example, why is Topology so central to the proof of such an elementary geometrical result? Or, how could someone unfamiliar with the Mobius transformation $1/z$ or geometric inversion deal with unbounded circular sets?

The first question can be answered by the observation that the Theorem is actually a result about closures of circular sets, rather than circular sets proper. This observation gains in credibility when one looks at the examples presented in the preceeding two sections: the complexities associated with those examples make a non-topological proof of the Theorem seem as improbable as a purely geometrical characterization of circularity.

As for the second question, let us point out that, due to the loss of boundedness, and therefore of compactness, unexpected difficulties arise in the study of unbounded circular sets. Therefore, unless it is possible to reduce the unbounded case to the bounded one, using the mapping $1/z$ for example, some new ideas and an almost certainly lengthier proof are required. We attempted to provide such a proof without success; our efforts are partially reflected on the following review diagram (Fig. 18).

Classifying circular sets: a topological view

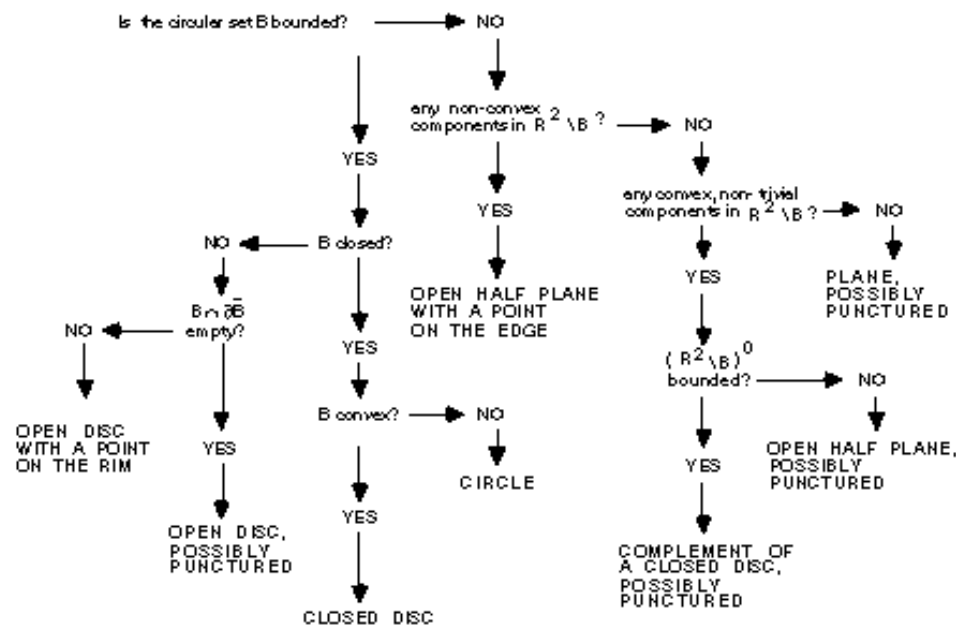


Fig. 18

Of course the left side of the diagram represents the way we studied the bounded case. The right side provides the reader with a major challenge: justify the arrows in the lower right corner! (Or, in different terms, show that the only way to preserve the circularity of the plane by drilling holes in it is to drill only one perfectly round hole, possibly of infinite radius.) In view of the following middle-of-the-diagram result, this would actually provide an alternative proof of the Theorem.

Exercise 15. Show that if the complement of an unbounded circular set B has a non-convex component, then B is an open half plane with one point on the edge.

To encourage the reader to search for alternative proofs of the Theorem, not necessarily dictated by our diagram, let us conclude by

stating our opinion that the shortest proof is not necessarily the most interesting.

Appendix 1: More on the Chord Lemma.

We introduced the Chord Lemma, in relation to Proposition 1, as a useful tool for the classification of circular sets. Here we suggest that the Chord Lemma could be important for its own sake, as a possible characterization of the closed disc. More specifically, we conjecture that the following converse of the Chord Lemma holds.

Conjecture. Let B be a compact, convex subset of $\mathbf{R}^2$ with $B^\circ \neq \emptyset$. Assume that there exists a non-maximum chord PQ of B with the following property: for every two points R, S of B° lying on opposite sides of PQ , every curve similar to ∂B that passes through R and S intersects PQ at a point of B° . Then B is a closed disc.

In simpler terms, our conjecture states the following: let PQ be a non-maximum chord of a compact, convex set $B \subset \mathbf{R}^2$ other than the closed disc; then there exist a set B' similar to B and two points $R, S \in B^\circ$ lying on opposite sides of PQ such that $\partial B'$ passes through R and S and does not intersect PQ at a point of B° (Fig. 19).

Two related results have been obtained in [11] as follows.

Proposition 5 [11]. Let B be a compact, convex subset of $\mathbf{R}^2$ with $B^\circ \neq \emptyset$. Assume that every chord PQ of B has the following property: for every two points R, S of B° lying on opposite sides of PQ , every curve similar to ∂B that passes through R and S intersects PQ at a point of B° . Then B is a closed disc.

Proposition 6 [11]. Let B be a compact, convex subset of $\mathbf{R}^2$ with $B^\circ \neq \emptyset$. Assume that every chord PQ of B has the following property: for every two points R, S of B° lying on opposite sides of PQ , every circle that passes through R and S intersects PQ at a point of B° . Then B is a closed disc.

Appendix 2: More on chords.

Proposition 2 characterizes the closed disc as the only compact, convex, circular set. In a first attempt to prove it we came across another characterization of the closed disc, which was subsequently verified and slightly strengthened by our colleague Steven Reyner as follows.

Proposition 7 [9]. If B is a compact, convex subset of $\mathbf{R}^2$ with $B^\circ \neq \emptyset$, then B is a closed disc if and only if it has the following property: there exists a point $M \in B^\circ$ such that all chords of B that pass through M are maximum.

Proposition 7 implies in particular that the "maximum chord" point M is unique; this brings us close to a famous unsolved problem.

The equichordal point problem [6]. Does there exist a convex, compact set B with two equichordal points? (A point $M \in B^0$ is called equichordal if all chords of B that pass through M are of equal (but not necessarily maximum) length.)

An example of a compact, convex set with an equichordal point (other than the closed disc) is $\{(r, \theta) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2 + \cos \theta\}$, which is known to every student of polar coordinates as a limaçon. It is easy to check that all chords through the pole $(0,0)$ have length equal to 4, while, for example, the chord through $(2, 3\pi/2)$ and $(2 + \sqrt{3}, \pi/3)$ has length approximately equal to 4.35. (It can be shown that the length of the diameter exceeds 4.4.)

Appendix 3: More on roundness.

We conclude with a potential application of circularity, trying to find a way for telling how "round" a convex, compact subset of $\mathbf{R}^2$ is. Obviously, a "round" set should have as few sharp corners as possible; and yet, although a rectangle and a square are identical with respect to their corners, it is intuitively clear that the square is more "round" than the rectangle. This leads to the idea that the closer a convex, compact set is to being a closed disc, the more

"round" it is. An important property of the closed disc which we studied in this work is that any two points can "see each other circularly", in the sense that there exists a circle passing through them that remains inside the closed disc; locally speaking, every point can "circularly see" everything within the set. Therefore, we can say that the further the average point of a set can "circularly see", the more "round" the set is; of course, how far a point "sees circularly" will have to be relative to the size of the set.

After this intuitive introduction, the following two definitions seem natural.

The circular range of $x \in B$ is defined by

$$V_r(x) = \{y: y \in B, \text{ there exists circle } C_y \subset B \text{ with } x, y \in C_y\}.$$

The roundness degree of a convex, compact $B \subset \mathbf{R}^2$ is defined by

$$r(B) = \text{average value, over } x \in B, \text{ of } \frac{m(V_r(x))}{m(B)}, \text{ where } m \text{ stands for}$$

planar Lebesgue measure.

Of course, these definitions apply as well to sets that may fail to be convex or compact.

It is clear now that the roundness degree of a square should exceed the roundness degree of a rectangle, although the computation of these degrees is not easy. For any convex, compact $B \subset \mathbf{R}^2$ such that $B^0 \neq \emptyset$, it is easy to see that $0 < r(B) \leq 1$, while $r(B) = 1$ holds if and only if B is a closed disc.

Rather than mentioning here a number of possible questions, some of which are stated in [2], we invite the readers to develop their own challenges, conjectures and computations.

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